On some inequalities of Hermite-Hadamard type via \( m \)-convexity

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**Abstract**

In this paper we give some estimates to the right-hand side of Hermite–Hadamard inequality for functions whose absolute values of second derivatives raised to positive real powers are \( m \)-convex. © 2010 Elsevier Ltd. All rights reserved.

1. Introduction

Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a convex function on the interval \( I \) of real numbers and \( a, b \in I \) with \( a < b \). If \( f \) is a convex function then the following double inequality, which is well known in the literature as the Hermite–Hadamard inequality, holds [1]:

\[
1 \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}.
\]

For recent results, generalizations and new inequalities related to the inequality presented above see [2–5]. In [6] G.Toader defined the concept of \( m \)-convexity as the following:

**Definition 1.** The function \( f : [0, b] \rightarrow \mathbb{R} \) is said to be \( m \)-convex, where \( m \in [0, 1] \), if for every \( x, y \in [0, b] \) and \( t \in [0, 1] \) we have:

\[
f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y).
\]

Denote by \( K_m(b) \) the set of the \( m \)-convex functions on \([0, b] \) for which \( f(0) \leq 0 \).

Some interesting and important inequalities for \( m \)-convex functions can be found in [7,8]. Using the classical results of Hermite and Hadamard on convex functions, S.S. Dragomir, P. Cerone and A. Sofo obtained the following result. (see [9]).

**Theorem 1.** Let \( f : [a, b] \rightarrow \mathbb{R} \) be a twice differentiable function on \((a, b)\) and suppose that \(-\infty < k \leq f''(x) \leq K < \infty\) for all \( x \in (a, b) \). Then the following inequality holds:

\[
k \frac{(b - a)^2}{12} \leq \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \leq K \frac{(b - a)^2}{12}.
\]
In order to prove our main results we need the following lemma (see [10]).

**Lemma 1.** Let \( f : I \subset \mathbb{R} \rightarrow \mathbb{R} \) be a twice differentiable mapping on \( I^o \), \( a, b \in I \) with \( a < b \) and \( f'' \in L[a, b] \). Then the following equality holds:

\[
\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx = \frac{(b-a)^2}{2} \int_0^1 t(1-t)f''(ta + (1-t)b)dt.
\]

The main purpose of this paper is to establish some new inequalities like those given in [10], but now for the class of \( m \)-convex functions.

2. Hermite–Hadamard type inequalities

We will start with the following theorem containing Hermite–Hadamard type inequality.

**Theorem 2.** Let \( f : I^o \rightarrow \mathbb{R} \), where \( I^o \subset [0, \infty) \) be a twice differentiable function on \( I^o \) such that \( f'' \in L[a, b] \), where \( a, b \in I \), \( a < b \). If \( |f''|^q \) is \( m \)-convex on \([a, b] \) for some fixed \( m \in (0, 1) \) and \( q \geq 1 \) then the following inequality holds:

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{2} \left[ \left( \frac{|f''(a)|^q + m |f''(b)|^q}{m} \right)^{\frac{1}{q}} \right].
\]

**Proof.** First suppose that \( q = 1 \). From Lemma 1 we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{2} \int_0^1 t(1-t) \left| f''(ta + (1-t)b) \right| dt.
\]

Since \( |f''| \) is \( m \)-convex on \([a, b] \) we know that for any \( t \in [0, 1] \)

\[
|f''(ta + (1-t)b)| \leq t |f''(a)| + m(1-t) \left| f'' \left( \frac{b}{m} \right) \right|.
\]

Therefore,

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{2} \int_0^1 t(1-t) \left( t |f''(a)| + m(1-t) \left| f'' \left( \frac{b}{m} \right) \right| \right) dt
\]

\[
= \frac{(b-a)^2}{2} \left[ \left( \frac{|f''(a)| + m |f''(b)|}{m} \right)^{\frac{1}{q}} \right].
\]

which completes the proof for this case.

Suppose now that \( q > 1 \). Using Lemma 1 and the well-known Hölder’s inequality (see for example [11]) for \( q, p = \frac{q}{q-1} \) we obtain

\[
\int_0^1 t(1-t) \left| f''(ta + (1-t)b) \right| dt = \int_0^1 (t-t^2)^{1-\frac{1}{q}} (t-t^2)^{\frac{1}{q}} \left| f''(ta + (1-t)b) \right| dt
\]

\[
\leq \left[ \int_0^1 (t-t^2)^{\frac{1}{q}} dt \right]^{\frac{q}{q-1}} \left[ \int_0^1 (t-t^2)^{\frac{1}{q}} \left| f''(ta + (1-t)b) \right|^{q} dt \right]^{\frac{1}{q}}.
\]

Since \( |f''|^q \) is \( m \)-convex on \([a, b] \) we know that for every \( t \in [0, 1] \)

\[
|f''(ta + (1-t)b)|^q \leq t |f''(a)|^q + m(1-t) \left| f'' \left( \frac{b}{m} \right) \right|^q.
\]

Hence, from (2.1) and (2.3) we obtain

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{2} \left[ \int_0^1 (t-t^2)^{\frac{1}{q}} dt \right]^{\frac{q}{q-1}} \left[ \int_0^1 (t-t^2)^{\frac{1}{q}} \left( t |f''(a)|^q + m(1-t) \left| f'' \left( \frac{b}{m} \right) \right|^q \right) dt \right]^{\frac{1}{q}}
\]
Let \( f \in L^1 \) if \( m \) is used to evaluate the integral

\[
\left( \frac{b-a}{2} \right)^{q-1} \left[ \left| f''(a) \right|^q + m \left| f''(b) \right|^q \right]^{\frac{1}{q}}
\]

\[
= \left( \frac{b-a}{2} \right)^{q-1} \left[ \left| f''(a) \right|^q + m \left| f''(b) \right|^q \right]^{\frac{1}{q}}
\]

which completes the proof. □

**Remark 1.** If in Theorem 2 we choose \( m = 1 \) and if \( |f''(x)| \leq K \) on \([a, b]\) we obtain

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{(b-a)^2}{12} \left\{ \left| f''(a) \right| + \left| f''(b) \right| \right\}
\]

\[
= K \frac{(b-a)^2}{12},
\]

which is the right-hand side of (1.1).

A similar result is embodied in the following theorem.

**Theorem 3.** Let \( f : I^q \to \mathbb{R} \), where \( I^q \subset [0, \infty) \) be a twice differentiable function on \( I^q \). \( a, b \in I \) with \( a < b \) and suppose that \( f'' \in L[a, b] \). If \( |f''|^q \) is \( m \)-convex on \([a, b]\) for some fixed \( q > 1 \) and \( m \in (0, 1] \) then the following inequality holds:

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{(b-a)^2}{12} \left\{ \left| f''(a) \right| + \left| f''(b) \right| \right\}
\]

where \( p = \frac{q}{q-1} \).

**Proof.** From Lemma 1 and using the well-known Hölder’s inequality we have successively

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{(b-a)^2}{2} \int_0^1 t(1-t) \left| f''(ta + (1-t)b) \right| \, dt
\]

\[
\leq \frac{(b-a)^2}{2} \left( \int_0^1 (1-t^2)^p \, dt \right)^{\frac{q}{p}} \left( \int_0^1 \left| f''(ta + (1-t)b) \right|^q \, dt \right)^{\frac{1}{q}}
\]

\[
\leq \frac{(b-a)^2}{2} \left( \int_0^1 (1-t^2)^p \, dt \right)^{\frac{1}{p}} \left( \int_0^1 f''(a)^q t \, dt + m \int_0^1 f''(b)^q (1-t) \, dt \right)^{\frac{1}{q}}
\]

\[
= \frac{(b-a)^2}{2} \left( 2^{-1-2p} \sqrt{\pi} \Gamma(1+p) \right)^{\frac{1}{p}} \left( \Gamma \left( \frac{3}{2} + p \right) \right)^{\frac{1}{q}} \left( \int_0^1 f''(a)^q \, dt + m \int_0^1 f''(b)^q \, dt \right)^{\frac{1}{q}}
\]

\[
= \frac{(b-a)^2}{2} \left( \frac{\pi^{\frac{1}{2}}}{2^2} \Gamma \left( \frac{3}{2} + p \right) \right)^{\frac{1}{q}} \left( \Gamma \left( \frac{1}{2} + p \right) \right)^{\frac{1}{p}} \left( \int_0^1 f''(a)^q + m \int_0^1 f''(b)^q \, dt \right)^{\frac{1}{q}}
\]

and since \( \sqrt{\pi} < 2 \), then we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{(b-a)^2}{8} \left( \Gamma \left( 1 + p \right) \right)^{\frac{1}{p}} \left( \Gamma \left( \frac{3}{2} + p \right) \right)^{\frac{1}{q}} \left( \int_0^1 f''(a)^q + m \int_0^1 f''(b)^q \, dt \right)^{\frac{1}{q}}
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).

We note that, the Beta and the Gamma function (see [12], pp 908–910).

\[
\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} \, dt, \quad x, y > 0, \quad \Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt, \quad x > 0,
\]

are used to evaluate the integral

\[
\int_0^1 (t-t^2)^p \, dt = \int_0^1 t^p (1-t)^p \, dt = \beta(p+1, p+1),
\]
where,
\[ \beta(x, x) = 2^{1-2\xi} \beta \left( \frac{1}{2}, x \right), \quad \text{and} \quad \beta(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}, \]

and we can obtain that
\[ \beta(p+1, p+1) = 2^{1-2(p+1)\xi} \beta \left( \frac{1}{2}, p+1 \right) = 2^{1-2(p+1)\xi} \frac{\Gamma \left( \frac{1}{2} \right) \Gamma(p+1)}{\Gamma \left( \frac{3}{2} + p \right)}, \]

and, \( \Gamma \left( \frac{1}{2} \right) \) = \( \sqrt{\pi} \), which completes the proof. \( \square \)

**Corollary 1.** With the above assumptions given that \( |f''(x)| \leq K \) on \([a, b]\), and \( 0 < m \leq 1 \), we have the inequality
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq K \left( \frac{b-a}{2} \right)^{\frac{1}{2}} \left( \frac{\Gamma(1+p)}{\Gamma \left( \frac{3}{2} + p \right)} \right)^{\frac{1}{2}}.
\]

**Corollary 2.** From Theorems 2 and 3 we have the inequality for \( q > 1 \),
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \min \{ K_1, K_2 \}
\]
where
\[
K_1 = \frac{(b-a)^2}{12} \left[ \left| f''(a) \right|^q + m \left| f'' \left( \frac{b}{m} \right) \right|^q \right]^{\frac{1}{q}},
\]
\[
K_2 = \frac{(b-a)^2}{8} \left( \frac{\Gamma(1+p)}{\Gamma \left( \frac{3}{2} + p \right)} \right)^{\frac{1}{2}} \left( \frac{\left| f''(a) \right|^q + m \left| f'' \left( \frac{b}{m} \right) \right|^q}{2} \right)^{\frac{1}{2}}.
\]

Another Hermite–Hadamard type inequality for powers in terms of the second derivatives is obtained as following:

**Theorem 4.** With the assumptions of Theorem 3 we have the inequality:
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{2} \left( \frac{\left| f''(a) \right|^q + m(q+1) \left| f'' \left( \frac{b}{m} \right) \right|^q}{2} \right)^{\frac{1}{q}}.
\]

**Proof.** From Lemma 1 and Hölder's inequality we obtain
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{2} \int_0^1 t(1-t) \left| f''(ta + (1-t)b) \right| dt
\]
\[
\leq \frac{(b-a)^2}{2} \left( \int_0^1 t^q dt \right)^{\frac{1}{q}} \left( \int_0^1 (1-t)^q \left| f''(ta + (1-t)b) \right|^q dt \right)^{\frac{1}{q}}
\]
\[
\leq \frac{(b-a)^2}{2} \left( \int_0^1 t^q dt \right)^{\frac{1}{q}} \left( \frac{\left| f''(a) \right|^q + m(1+q) \left| f'' \left( \frac{b}{m} \right) \right|^q}{(q+1)(q+2)} \right)^{\frac{1}{q}}.
\]

Since
\[
\lim_{p \to x^+} \left( \frac{1}{1+p} \right)^{\frac{1}{p}} = 1 \quad \text{and} \quad \lim_{p \to x^-} \left( \frac{1}{1+p} \right)^{\frac{1}{p}} = \frac{1}{2},
\]
we have
\[
\frac{1}{2} < \left( \frac{1}{1+p} \right)^{\frac{1}{p}} < 1, \quad q \in (1, \infty),
\]
hence for \( q \in (1, \infty) \)
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{2} \left( \frac{\left| f''(a) \right|^q + m(q+1) \left| f'' \left( \frac{b}{m} \right) \right|^q}{2} \right)^{\frac{1}{q}}. \quad \square
\]
The following result also holds:

**Theorem 5.** Let $f : I^o \to \mathbb{R}$, where $I^o \subset [0, \infty)$ be a twice differentiable function on $I^o$. If $f''(1)$ is m-convex on $[a, b]$ for some fixed $q \geq 1$ and $m \in (0, 1)$ then the following inequality holds:

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) dx\right| \leq \frac{(b - a)^2}{4} \left(2 \left|f''(a)\right|^q + m(q + 1) \left|f''\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}.$$

**Proof.** From Lemma 1 and the well-known power-mean inequality we obtain

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) dx\right| \leq \frac{(b - a)^2}{2} \left(\int_0^1 t(1 - t) \left|f''(ta + (1 - t)b)\right| dt\right)$$

$$\leq \frac{(b - a)^2}{2} \left(\int_0^1 t(1 - t)^q \left|f''(ta + (1 - t)b)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{(b - a)^2}{2} \left(\int_0^1 t(1 - t)^q \left[\left|f''(a)\right|^q t + \left|f''\left(\frac{b}{m}\right)\right|^q m(1 - t) dt\right]\right)^{\frac{1}{q}}$$

$$\leq \frac{(b - a)^2}{2} \left(\int_0^1 t(1 - t)^q \left[\left|f''(a)\right|^q t + \left|f''\left(\frac{b}{m}\right)\right|^q m(1 - t) dt\right]\right)^{\frac{1}{q}}$$

$$= \frac{(b - a)^2}{2} \left(\frac{1}{2} \right) \left[\left|f''(a)\right|^q + m \left|f''\left(\frac{b}{m}\right)\right|^q \right]^{\frac{1}{q}}$$

$$= \frac{(b - a)^2}{4} \left(\frac{2}{(q + 1)(q + 2)(q + 3)}\right)^{\frac{1}{q}} \left[2 \left|f''(a)\right|^q + m(q + 1) \left|f''\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}.$$

Since \(\left(\frac{2}{(q + 1)(q + 2)(q + 3)}\right)^{\frac{1}{q}} \leq 1, q \in [1, \infty),\) we obtain

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) dx\right| \leq \frac{(b - a)^2}{4} \left[2 \left|f''(a)\right|^q + m(q + 1) \left|f''\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}$$

which completes the proof. \(\square\)

**Remark 2.** From Theorems 3–5, we have

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) dx\right| \leq \min\{E_1, E_2, E_3\}$$

where

$$E_1 = \frac{(b - a)^2}{8} \left(\frac{f'(1 + p)}{f'(\frac{1}{2} + p)}\right)^{\frac{1}{q}} \left[\left|f''(a)\right|^q + m \left|f''\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}$$

$$E_2 = \frac{(b - a)^2}{2} \left(\left|f''(a)\right|^q + m(q + 1) \left|f''\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}$$

$$E_3 = \frac{(b - a)^2}{4} \left(2 \left|f''(a)\right|^q + m(q + 1) \left|f''\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}.$$

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