A NOTE ON THE NONLOCAL BOUNDARY VALUE PROBLEM FOR HYPERBOLIC-PARABOLIC DIFFERENTIAL EQUATIONS

ALLABEREN ASHYRALYEV∗ AND YILDIRIM OZDEMIR†

Abstract. The nonlocal boundary value problem
\[
\begin{aligned}
\frac{d^2 u(t)}{dt^2} + Au(t) &= f(t)(0 \leq t \leq 1), \\
\frac{du(t)}{dt} + Au(t) &= g(t)(-1 \leq t \leq 0), \\
u(-1) &= \alpha u(\mu) + \beta u'(\lambda) + \varphi, \quad |\alpha|, |\beta| \leq 1, 0 < \mu, \lambda \leq 1
\end{aligned}
\]
for differential equation in a Hilbert space $H$ with the self-adjoint positive definite operator $A$ is considered. The stability estimates for the solution of this problem are established. In applications, the stability estimates for the solutions of the mixed type boundary value problems for hyperbolic-parabolic equations are obtained.

Key Words. hyperbolic-parabolic equation, nonlocal boundary value problem, stability.

AMS(MOS) subject classification. 65N, 65J, 47D, 34G, 35M

1. Introduction. It is known that some problems in fluid mechanics (model of the motion of an ideal fluid filling, exhibiting both viscous and nonviscous phases) and other areas of physics and mathematical biology (taxis-diffusion-reaction model) lead to partial differential equations of the hyperbolic-parabolic type. Methods of solutions of the nonlocal boundary value problems for hyperbolic-parabolic differential equations have been studied extensively by many researches (see, e.g., [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12] and the references given therein).

In the present paper we consider the nonlocal boundary value problem
\[
\begin{aligned}
\frac{d^2 u(t)}{dt^2} + Au(t) &= f(t)(0 \leq t \leq 1), \\
\frac{du(t)}{dt} + Au(t) &= g(t)(-1 \leq t \leq 0), \\
u(-1) &= \alpha u(\mu) + \beta u'(\lambda) + \varphi, \quad |\alpha|, |\beta| \leq 1, 0 < \mu, \lambda \leq 1
\end{aligned}
\]
for differential equations of mixed type in a Hilbert space $H$ with self-adjoint positive definite operator $A$.

∗ Department of Mathematics, Fatih University, Istanbul, Turkey and International Turkmen-Turkish University, Ashgabat, Turkmenistan
† Department of Mathematics, Fatih University, Istanbul, Turkey
A function $u(t)$ is called a solution of the problem (1.1) if the following conditions are satisfied:

i. $u(t)$ is twice continuously differentiable on the interval $(0,1]$ and continuously differentiable on the segment $[-1,1]$. The derivative at the endpoints of the segment are understood as the appropriate unilateral derivatives.

ii. The element $u(t)$ belongs to $D(A)$ for all $t \in [-1,1]$, and the function $Au(t)$ is continuous on the segment $[-1,1]$.

iii. $u(t)$ satisfies the equations and nonlocal boundary condition (1.1).

In the paper [18] the following theorem on the stability was proved.

**Theorem 1.1.** Suppose that $\varphi \in D(A)$ and $f(t)$ be continuously differentiable on $[0,1]$ and $g(t)$ be continuously differentiable on $[-1,0]$ functions and $\beta = 0$. Then there is a unique solution of the problem (1.1) and the stability inequalities

$$\max_{-1 \leq t \leq 1} \| u(t) \|_H \leq M[\| \varphi \|_H$$

$$+ \max_{-1 \leq t \leq 0} \| g(t) \|_H + \max_{0 \leq t \leq 1} \| A^{-1/2}f(t) \|_H],$$

$$\max_{-1 \leq t \leq 1} \| A^{1/2}u(t) \|_H \leq M[\| A^{1/2}\varphi \|_H$$

$$+ \| g(0) \|_H + \int_{-1}^{0} \| g'(t) \|_H dt + \max_{0 \leq t \leq 1} \| f(t) \|_H],$$

$$\max_{-1 \leq t \leq 0} \left\| \frac{du(t)}{dt} \right\|_H + \max_{0 \leq t \leq 1} \left\| \frac{d^2u(t)}{dt^2} \right\|_H$$

$$+ \max_{-1 \leq t \leq 1} \| Au(t) \|_H \leq M[\| A\varphi \|_H + \| A^{1/2}g(0) \|_H$$

$$+ \| f(0) \|_H + \max_{-1 \leq t \leq 0} \| g'(t) \|_H + \int_{0}^{1} \| f'(t) \|_H dt]$$

hold, where $M$ does not depend on $f(t), g(t)$, and $\varphi$. 
We are interested in studying the stability of solutions of the problem (1.1) for $\beta \neq 0$. We have not been able to obtain the same stability estimates for the solutions of the problem (1.1) for $\beta \neq 0$. Nevertheless, in the present paper the stability estimates for the solution of the problem (1.1) under a stronger assumption than $f(t)$ be continuously differentiable on $[0, 1]$ and $g(t)$ be continuously differentiable on $[-1, 0]$ functions are established. In applications, the stability estimates for the solutions of the of the mixed type boundary value problems for the hyperbolic-parabolic equations are obtained.

Finally note that the methods for numerical solutions of the nonlocal boundary value problem (1.1) in the case $\beta = 0$ have been studied extensively (see [14] - [17], [19] - [21]and the references therein).

2. The main theorem. First of all let us give some lemmas that will be needed below.

Lemma 2.1. The estimates hold:

\begin{equation}
\|c(t)\|_{H \rightarrow H} \leq 1, \quad \|A^{1/2}s(t)\|_{H \rightarrow H} \leq 1,
\end{equation}

\begin{equation}
\|A^\gamma e^{-tA}\|_{H \rightarrow H} \leq Mt^{-\gamma}e^{-\delta t}, \quad t > 0, \quad 0 \leq \gamma \leq 1, \quad \delta > 0, \quad M > 0,
\end{equation}

where

\[c(t) = \frac{e^{itA^{1/2}} + e^{-itA^{1/2}}}{2}, \quad s(t) = A^{-1/2}\frac{e^{itA^{1/2}} - e^{-itA^{1/2}}}{2i}.\]

Lemma 2.2. The operator

\[I - \alpha [c(\mu) - As(\mu)] e^{-A} + \beta [s(\lambda) + c(\lambda)] Ae^{-A}\]

has an inverse

\[T = (I - \alpha [c(\mu) - As(\mu)] e^{-A} + \beta [s(\lambda) + c(\lambda)] Ae^{-A})^{-1}\]

and the estimate holds:

\begin{equation}
\|T\|_{H \rightarrow H} \leq M.
\end{equation}

Proof. The proof of the estimate (2.3) is based on the estimate

\[\| - \alpha [c(\mu) - As(\mu)] e^{-A} + \beta [s(\lambda) + c(\lambda)] Ae^{-A}\|_{H \rightarrow H} < 1.\]
Using the definitions of \( c(\mu) \) and \( s(\mu) \) and positivity and self-adjointness property of \( A \), we obtain

\[
\left\| -\alpha [c(\mu) - As(\mu)] e^{-A} + \beta [s(\lambda) + c(\lambda)] Ae^{-A} \right\|_{H \rightarrow H} \leq \sup_{\delta \leq \rho < \infty} \left| -\alpha \left[ \cos(\sqrt{\rho}\mu) - \sqrt{\rho} \sin(\sqrt{\rho}\mu) \right] e^{-\rho} \right.
\]

\[
+ \beta \left[ \sqrt{\rho} \sin(\sqrt{\rho}\lambda) + \rho \cos(\sqrt{\rho}\lambda) \right] e^{-\rho}.
\]

Since

\[
\cos(\sqrt{\rho}\mu) - \sqrt{\rho} \sin(\sqrt{\rho}\mu) = \sqrt{\rho + 1} \cos(\sqrt{\rho}\mu - \mu_0),
\]

\[
\sqrt{\rho} \sin(\sqrt{\rho}\lambda) + \rho \cos(\sqrt{\rho}\lambda) = \sqrt{\rho} \sqrt{\rho + 1} \cos(\sqrt{\rho}\mu - \mu_1),
\]

we have that

\[
\left\| -\alpha [c(\mu) - As(\mu)] e^{-A} + \beta [s(\lambda) + c(\lambda)] Ae^{-A} \right\|_{H \rightarrow H} \leq \sup_{\delta \leq \rho < \infty} \sqrt{\rho + 1} (1 + \sqrt{\rho}) e^{-\rho}.
\]

It is easy to show that \( \sup_{\delta \leq \rho < \infty} \sqrt{\rho + 1} (1 + \sqrt{\rho}) e^{-\rho} < 1 \). Thus Lemma 2.2 is proved.

Now, we will obtain the formula for solution of the problem (1.1). It is known that for smooth data of the initial value problems

(2.4) \[
\begin{cases}
    u''(t) + Au(t) = f(t), & (0 \leq t \leq 1),
    
    u(0) = u_0, \ u'(0) = u'_0,
\end{cases}
\]

(2.5) \[
\begin{cases}
    u'(t) + Au(t) = g(t), & (-1 \leq t \leq 0),
    
    u(-1) = u_{-1},
\end{cases}
\]

there are unique solutions of the problems (2.4), (2.5), and the following formulas hold:

(2.6) \[
u(t) = c(t) u(0) + s(t) u'(0) + \int_0^t s(t - y) f(y) dy, \quad 0 \leq t \leq 1,
\]
and

\[ u(t) = e^{-(t+1)A}u_{-1} + \int_{-1}^{t} e^{-(t-y)A}g(y)\,dy, \quad -1 \leq t \leq 0. \quad (2.7) \]

Using formulas (2.6), (2.7), and equation (1.1) we can write

\[ u(t) = [c(t) - As(t)] \left\{ e^{-A}u_{-1} + \int_{-1}^{0} e^{yA}g(y)\,dy \right\} \]

\[ + s(t)g(0) + \int_{0}^{t} s(t-y)f(y)\,dy. \quad (2.8) \]

Now, using the condition \( u(-1) = \alpha u(\mu) + \beta u'(\lambda) + \varphi \), we obtain the operator equation

\[ \{ I - \alpha [c(\mu) - As(\mu)] e^{-A} + \beta [s(\lambda) + c(\lambda)] Ae^{-A} \} u_{-1} \]

\[ = \alpha \left\{ c(\mu) \int_{-1}^{0} e^{yA}g(y)\,dy + s(\mu) \left[ g(0) - A \int_{-1}^{0} e^{yA}g(y)\,dy \right] \right\} \]

\[ + \int_{0}^{\mu} s(\mu-y)f(y)\,dy \right\} + \beta \left\{ -As(\lambda) \int_{-1}^{0} e^{yA}g(y)\,dy \right\} \]

\[ + \beta \left\{ c(\lambda) \left[ g(0) - A \int_{-1}^{0} e^{yA}g(y)\,dy \right] + \int_{0}^{\lambda} c(\lambda-y)f(y)\,dy \right\} + \varphi. \]

Since the operator

\[ I - \alpha [c(\mu) - As(\mu)] e^{-A} + \beta [s(\lambda) + c(\lambda)] Ae^{-A} \]

has an inverse

\[ T = \left( I - \alpha [c(\mu) - As(\mu)] e^{-A} + \beta [s(\lambda) + c(\lambda)] Ae^{-A} \right)^{-1}, \]

for the solution of the operator equation (2.9) we have the formula

\[ u_{-1} = T \left[ \alpha \left\{ c(\mu) \int_{-1}^{0} e^{yA}g(y)\,dy \right\} \right. \]

\[ \left. + \int_{0}^{\mu} s(\mu-y)f(y)\,dy \right\} + \beta \left\{ -As(\lambda) \int_{-1}^{0} e^{yA}g(y)\,dy \right\} \]

\[ + \beta \left\{ c(\lambda) \left[ g(0) - A \int_{-1}^{0} e^{yA}g(y)\,dy \right] + \int_{0}^{\lambda} c(\lambda-y)f(y)\,dy \right\} + \varphi. \]

\[ (2.10) \]
\[
+ s(\mu) \left[ g(0) - A \int_{-1}^{0} e^{yA} g(y) \, dy + \int_{0}^{\mu} s(\mu - y) f(y) \, dy \right] \\
+ \beta \left\{ -As(\lambda) \int_{-1}^{0} e^{yA} g(y) \, dy + c(\lambda) \left[ g(0) - A \int_{-1}^{0} e^{yA} g(y) \, dy + \int_{0}^{\lambda} c(\lambda - y) f(y) \, dy \right] + \varphi \right\}.
\]

Hence, for the solution of the nonlocal boundary value problem (1.1) we have the formulas (2.7), (2.8) and (2.10).

**Theorem 2.1.** Suppose that \( \varphi \in D(A) \), \( g(0) \in D(A^{1/2}) \), \( g'(0) \in H \), \( f(0) \in D(A^{1/2}) \) and \( f'(0) \in H \). Let \( f(t) \) be twice continuously differentiable on \([0, 1]\) and \( g(t) \) be twice continuously differentiable on \([-1, 0]\) functions. Then there is a unique solution of the problem (1.1) and the stability inequalities hold:

\[
\max_{-1 \leq t \leq 1} \| u(t) \|_{H} \leq M \left[ \| \varphi \|_{H} + \max_{-1 \leq t \leq 0} \| A^{1/2}g'(t) \|_{H} \right. \\
+ \| A^{1/2}g(0) \|_{H} + \| A^{1/2}f(0) \|_{H} + \max_{0 \leq t \leq 1} \| A^{1/2}f'(t) \|_{H} \right],
\]

\[
\max_{-1 \leq t \leq 1} \left\| \frac{du}{dt} \right\|_{H} + \max_{-1 \leq t \leq 1} \| A^{1/2}u(t) \|_{H} \leq M \left[ \| A^{1/2}\varphi \|_{H} \\
+ \| g(0) \|_{H} + \max_{-1 \leq t \leq 0} \| g'(t) \|_{H} + \| f(0) \|_{H} + \max_{0 \leq t \leq 1} \| f'(t) \|_{H} \right].
\]

\[
\max_{-1 \leq t \leq 0} \left\| \frac{du}{dt} \right\|_{H} + \max_{0 \leq t \leq 1} \left\| \frac{d^2u}{dt^2} \right\|_{H} + \max_{-1 \leq t \leq 1} \| Au(t) \|_{H} \leq M \left[ \| A\varphi \|_{H} + \| A^{1/2}g(0) \|_{H} + \| g'(0) \|_{H} + \max_{-1 \leq t \leq 0} \| g''(t) \|_{H} \\
+ \| A^{1/2}f(0) \|_{H} + \| f'(0) \|_{H} + \max_{0 \leq t \leq 1} \| f''(t) \|_{H} \right].
\]
where $M$ does not depend on $f(t)$, $t \in [0, 1]$, $g(t)$, $t \in [-1, 0]$ and $\varphi$.

**Proof.** First, we obtain estimate (2.11). Using formula (2.10) and an integration by parts, we obtain

\begin{equation}
(2.14)
\begin{align*}
u_{-1} &= T \left[ \alpha \left\{ c(\mu) \left[ A^{-1} \left( g(0) - e^{-A} g(-1) \right) ight. \right. \\
&\left. \left. - \int_{-1}^{0} e^{yA} g'(y) \, dy \right]\right] + s(\mu) \left( e^{-A} g(-1) + \int_{-1}^{0} e^{yA} g'(y) \, dy \right) \right. \\
&\left. + A^{-1} \left\{ f(\mu) - c(\mu) f(0) - \int_{0}^{\mu} c(\mu - y) f'(y) \, dy \right\} \right) \\
&\left. + \beta \left\{ -s(\lambda) \left( g(0) - e^{-A} g(-1) - \int_{-1}^{0} e^{yA} g'(y) \, dy \right) \right. \\
&\left. + c(\lambda) \left( e^{-A} g(-1) + \int_{-1}^{0} e^{yA} g'(y) \, dy \right) \\
&\left. + s(\lambda) f(0) + \int_{0}^{\lambda} s(\lambda - y) f'(y) \, dy \right\} + \varphi \right].
\end{align*}
\end{equation}

Using estimates (2.3), (2.1) and (2.2), we obtain

\begin{equation}
(2.15)
\begin{align*}
\| u_{-1} \|_H &\leq M \left[ \| \varphi \|_H + \max_{-1 \leq t \leq 0} \| A^{-1/2} g'(t) \|_H \\
&+ \| A^{-1/2} g(0) \|_H + \| A^{-1/2} f(0) \|_H + \max_{0 \leq t \leq 1} \| A^{-1/2} f'(t) \|_H \right].
\end{align*}
\end{equation}

Using formulas (2.7), (2.8) and an integration by parts, we obtain

\begin{equation}
(2.16)
\begin{align*}
u(t) &= e^{-(t+1)A} u_{-1} + A^{-1} \left( g(t) - e^{-A} g(-1) \\
&- \int_{-1}^{t} e^{yA} g'(y) \, dy, \quad -1 \leq t \leq 0, \right)
\end{align*}
\end{equation}

\begin{equation}
(2.17)
\begin{align*}
u(t) &= \left[ c(t) - As(t) \right] \left\{ e^{-A} u_{-1}
\right. \end{align*}
\end{equation}
\[ +A^{-1} \left( g(0) - e^{-A}g(-1) - \int_{-1}^{0} e^{yA}g'(y) \, dy \right) + s(t)g(0) \]

\[ +A^{-1} \left[ f(t) - c(t)f(0) - \int_{0}^{t} c(t-y)f'(y) \, dy, \quad 0 \leq t \leq 1. \right] \]

Using estimates (2.1) and (2.2), we obtain

\[ \| u(t) \|_H \leq M \left[ \| u_{-1} \|_H + \max_{-1 \leq t \leq 0} \| A^{-1/2}g'(t) \|_H + \| A^{-1/2}g(0) \|_H \right], \quad -1 \leq t \leq 0, \]

\[ \| u(t) \|_H \leq M \left[ \max_{0 \leq t \leq 1} \| A^{-1/2}f'(t) \|_H + \| A^{-1/2}f(0) \|_H \right. \]

\[ + \| u_{-1} \|_H + \max_{-1 \leq t \leq 0} \| A^{-1/2}g'(t) \|_H + \| A^{-1/2}g(0) \|_H \left. \right], \quad 0 \leq t \leq 1. \]

Then from (2.15) and the last two estimates, it follows (2.11).

Second, we obtain estimate (2.12). Applying \( A^{1/2} \) to the formula (2.14) and using estimates (2.3), (2.1) and (2.2), we obtain

\[ \| A^{1/2}u_{-1} \|_H \leq M \left[ \| A^{1/2} \varphi \|_H + \max_{-1 \leq t \leq 0} \| g'(t) \|_H \right. \]

\[ + \| g(0) \|_H + \| f(0) \|_H + \max_{0 \leq t \leq 1} \| f'(t) \|_H \left. \right] \]  \quad (2.18)

Applying \( A^{1/2} \) to the formulas (2.16), (2.17) and using estimates (2.1) and (2.2), we obtain

\[ \| A^{1/2}u(t) \|_H \leq M \left[ \| A^{1/2}u_{-1} \|_H + \max_{-1 \leq t \leq 0} \| g'(t) \|_H + \| g(0) \|_H \right], \quad -1 \leq t \leq 0, \]

\[ \| A^{1/2}u(t) \|_H \leq M \left[ \max_{0 \leq t \leq 1} \| f'(t) \|_H + \| f(0) \|_H \right. \]

\[ + \| A^{1/2}u_{-1} \|_H + \max_{-1 \leq t \leq 0} \| g'(t) \|_H + \| g(0) \|_H \right. \left. \right], \quad 0 \leq t \leq 1. \]
Then from (2.18) and the last two estimates, it follows (2.12).

Third, we obtain estimate (2.13). Using formula (2.14) and an integration by parts, we obtain

\[ u_{-1} = T \left[ \alpha \left\{ c(\mu) \left[ A^{-1} (g(0) - e^{-A}g(-1)) + A^{-1} \left[ g'(0) - e^{-A}g'(-1) - \int_{-1}^{0} e^{yA}g''(y) \, dy \right] \right] \right\} + s(\mu) \left( e^{-A}g(-1) + A^{-1} \left[ g'(0) - e^{-A}g'(-1) - \int_{-1}^{0} e^{yA}g''(y) \, dy \right] \right) + A^{-1} \left( f(\mu) - c(\mu) \left( g'(0) + \int_{0}^{\mu} s(\mu - y) f''(y) \, dy \right) \right) \right] + \beta \left( -s(\lambda) \left( e^{-A}g(-1) + A^{-1} \left[ g'(0) - e^{-A}g'(-1) - \int_{-1}^{0} e^{yA}g''(y) \, dy \right] \right) \right) + c(\lambda) \left( e^{-A}g(-1) + A^{-1} \left[ g'(0) - e^{-A}g'(-1) - \int_{-1}^{0} e^{yA}g''(y) \, dy \right] \right) + s(\lambda) f(0) + A^{-1} \left( f'(\lambda) - c(\lambda) f'(0) - \int_{0}^{\lambda} c(\lambda - y) f''(y) \, dy \right) + \varphi \right]. \]

Using estimates (2.3), (2.1) and (2.2), we obtain

\[
\| Au_{-1} \|_H \leq M \left[ \| A\varphi \|_H + \max_{-1 \leq t \leq 0} \| g''(t) \|_H + \| g'(0) \|_H \right] + \left[ \left\| A^{1/2}g(0) \right\|_H + \left\| A^{1/2}f(0) \right\|_H + \left\| f'(0) \right\|_H + \max_{0 \leq t \leq 1} \| f''(t) \|_H \right].
\]

Using formulas (2.16), (2.17) and an integration by parts, we obtain

\[ u(t) = e^{-(t+1)A}u_{-1} + A^{-1} \left( g(t) - e^{-A}g(-1) + A^{-1} \left[ g'(t) - e^{-A}g'(-1) - \int_{-1}^{t} e^{yA}g''(y) \, dy \right] \right), \quad -1 \leq t \leq 0,
\]
\[ u(t) = [c(t) - As(t)] \left\{ e^{-A}u_{-1} + A^{-1} \left( g(0) - e^{-A}g(-1) \right) \right. \]

\[ -A^{-1} \left[ g'(0) - e^{-A}g'(-1) - \int_{-1}^{0} e^{A}g''(y) \, dy \right] \} \]

\[ + s(t) g(0) + A^{-1} f(t) - c(t) f(0) \]

\[ - \left[ s(t) f'(0) + \int_{0}^{t} s(t - y) f''(y) \, dy \right], \quad 0 \leq t \leq 1. \]

Applying \( A \) to the last two formulas and using estimates (2.1) and (2.2), we obtain

\[ \|Au(t)\|_H \leq M \left[ \|Au_{-1}\|_H + \max_{-1 \leq t \leq 0} \|g''(t)\|_H + \|A^{1/2}g(0)\|_H + \|g'(0)\|_H \right], \quad -1 \leq t \leq 0, \]

\[ \|Au(t)\|_H \leq M \left[ \max_{0 \leq t \leq 1} \|f''(t)\|_H + \|A^{1/2}f(0)\|_H + \|f'(0)\|_H \right] + \|Au_{-1}\|_H + \max_{-1 \leq t \leq 0} \|g''(t)\|_H + \|A^{1/2}g(0)\|_H + \|g'(0)\|_H \], \quad 0 \leq t \leq 1. \]

Then from (2.19) and the last two estimates, it follows (2.13). Theorem 2.1 is proved.

**Remark 1.** Theorem 2.1 holds for the following multi-point nonlocal boundary value problem

\[
\begin{aligned}
\frac{d^2 u(t)}{dt^2} &+ Au(t) = f(t)(0 \leq t \leq 1), \\
\frac{du(t)}{dt} &+ Au(t) = g(t)(-1 \leq t \leq 0), \\
u(-1) &= \sum_{i=1}^{N} \alpha_i u(\mu_i) + \sum_{i=1}^{L} \beta_i u'(\lambda_i) + \varphi, \\
\sum_{i=1}^{N} |\alpha_i|, &\sum_{i=1}^{L} |\beta_i| \leq 1, \\
0 < \mu_i \leq 1, &1 \leq i \leq N, 0 < \lambda_i \leq 1, 1 \leq i \leq L
\end{aligned}
\]
in a Hilbert space $H$ with self-adjoint positive definite operator $A$.

**Remark 2.** We cannot obtain the stability estimates for the solution of the problem (1.1) in an arbitrary Banach space $E$ with strongly positive operator $A$ under the assumptions

\[(2.20) \quad \|c(t)\|_{E\to E} \leq M, \|A^{1/2}s(t)\|_{E\to E} \leq M.\]

Nevertheless, the nonlocal boundary value problem (1.1) generated by the following well-posed problem

\[(2.21) \quad \begin{cases} u''(t) + Au(t) = f(t) (0 \leq t \leq 1), \\ u'(t) + Au(t) = g(t) (-1 \leq t \leq 0), u(-1) = \varphi \end{cases}\]

for differential equations of mixed type in a Hilbert space $H$ with self-adjoint positive definite operator $A$, and ill-posed problem

\[(2.22) \quad \begin{cases} u''(t) + Au(t) = f(t) (0 \leq t \leq 1), \\ \alpha u(\mu) + \beta u'(\lambda) = \varphi, |\alpha|, |\beta| \leq 1, 0 < \mu, \lambda \leq 1, \\ u'(t) + Au(t) = g(t) (-1 \leq t \leq 0) \end{cases}\]

for differential equations of mixed type in a Hilbert space $H$ with self-adjoint positive definite operator $A$.

The stability estimates for the solution of the problem (2.21) in an arbitrary Banach space $E$ with strongly positive operator $A$ under the assumptions (2.20) can be established.

3. Applications. First, we consider the mixed problem for hyperbolic-parabolic equation

\[(3.1) \quad \begin{cases} v_{yy} - (a(x)v_x)_x + \delta v = f(y,x), 0 < y < 1, 0 < x < 1, \\ v_y - (a(x)v_x)_x + \delta v = g(y,x), -1 < y < 0, 0 < x < 1, \\ v(-1,x) = v(1,x) + v_y(1,x) + \varphi(x), 0 \leq x \leq 1, \\ v(y,0) = v(y,1), v_x(y,0) = v_x(y,1), -1 \leq y \leq 1, \\ v(0+, x) = v(0-, x), v_y(0+, x) = v_y(0-, x), 0 \leq x \leq 1. \end{cases}\]

Problem (3.1) has a unique smooth solution $v(y,x)$ for the smooth $a(x) > 0(x \in (0, 1))$, $\varphi(x)$ ($x \in [0, 1]$) and $f(y,x)(y \in [0, 1], x \in [0, 1])$, $g(y,x)(y \in [-1, 0], x \in [0, 1])$ functions and $\delta = \text{const} > 0$. This allows us to reduce the mixed problem (3.1) to the nonlocal boundary value problem (1.1) in Hilbert space $H$ with a self-adjoint positive definite operator $A$ defined by (3.1). Let us give a number of corollaries of the abstract Theorem 2.1.
Theorem 3.1. The solutions of the nonlocal boundary value problem (3.1) satisfy the stability estimates

\[
\max_{-1 \leq y \leq 1} \|v(y)\|_{L^2[0,1]} \leq M \left[ \|f(0)\|_{L^2[0,1]} + \max_{0 \leq y \leq 1} \|f_y(y)\|_{L^2[0,1]} + \|g(0)\|_{L^2[0,1]} + \max_{-1 \leq y \leq 0} \|g_y(y)\|_{L^2[0,1]} + \|\varphi\|_{L^2[0,1]} \right],
\]

\[
\max_{-1 \leq y \leq 1} \|v(y)\|_{W^2_2[0,1]} \leq M \left[ \|f(0)\|_{L^2[0,1]} + \max_{0 \leq y \leq 1} \|f_y(y)\|_{L^2[0,1]} + \|g(0)\|_{L^2[0,1]} + \max_{-1 \leq y \leq 0} \|g_y(y)\|_{L^2[0,1]} + \|\varphi\|_{W^2_2[0,1]} \right],
\]

\[
\max_{-1 \leq y \leq 1} \|v(y)\|_{W^2_2[0,1]} + \max_{-1 \leq y \leq 0} \|v_y(y)\|_{L^2[0,1]} + \max_{0 \leq y \leq 1} \|v_{yy}(y)\|_{L^2[0,1]} \leq M \left[ \|\varphi\|_{W^2_2[0,1]} + \|f(0)\|_{W^2_2[0,1]} + \|f_y(0)\|_{L^2[0,1]} + \max_{0 \leq y \leq 1} \|f_{yy}(y)\|_{L^2[0,1]} \right]
\]

where \(M\) does not depend on \(f(y,x)\) (\(y \in [0,1], x \in [0,1]\)), \(g(y,x)\) (\(y \in [-1,0], x \in [0,1]\)) and \(\varphi(x)\) (\(x \in [0,1]\)).

The proof of this theorem is based on the abstract Theorem 2.1 and the symmetry properties of the space operator generated by the problem (3.1).

Second, let \(\Omega\) be the unit open cube in the n-dimensional Euclidean space \(\mathbb{R}^n\) \((0 < x_k < 1, \ 1 \leq k \leq n)\) with boundary \(S, \ \overline{\Omega} = \Omega \cup S\). In \([0,1] \times \Omega\) we consider the mixed boundary value problem for the multidimensional hyperbolic-parabolic equation

\[
\begin{align*}
    v_{yy} - \sum_{r=1}^{n} (a_r(x)v_{x_r})_{x_r} &= f(y,x), \quad 0 \leq y \leq 1, \\
    v_{y} - \sum_{r=1}^{n} (a_r(x)v_{x_r})_{x_r} &= g(y,x), \quad -1 \leq y \leq 0, \\
    v(-1, x) &= v(1, x) + v_y(1, x) + \varphi(x), \quad x \in \overline{\Omega}, \\
    u(y,x) &= 0, \quad x \in S, \quad -1 \leq y \leq 1, \\
    v(0+, x) &= v(0-, x), \quad v_y(0+, x) = v_y(0-, x), \quad x \in \overline{\Omega}.
\end{align*}
\]

(3.2)
We introduce the Hilbert spaces $L^2(\Omega)$ of the all integrable functions defined on $\Omega$, equipped with the norm
\[
\|f\|_{L^2(\Omega)} = \left\{ \int \cdots \int_{x \in \Omega} |f(x)|^2 dx_1 \cdots dx_n \right\}^{1/2}.
\]

Problem (3.2) has a unique smooth solution $v(y, x)$ for the smooth $a_r(x) \geq \delta > 0$, $\varphi(x)$ $(x \in \Omega)$ and $f(y, x)$ $(y \in (0, 1), x \in \Omega), g(y, x)$ $(y \in (-1, 0), x \in \Omega)$ functions. This allows us to reduce the mixed problem (3.2) to the non-local boundary value problem (1.1) in Hilbert space $H$ with a self-adjoint positive definite operator $A$ defined by (3.2). Let us give a number of corollaries of the abstract Theorem 2.1.

**Theorem 3.2.** The solutions of the nonlocal boundary value problem (3.2) satisfy the stability estimates

\[
\max_{-1 \leq y \leq 1} \|v(y)\|_{L^2(\Omega)} \leq M \left[ \|f(0)\|_{L^2(\Omega)} + \max_{0 \leq y \leq 1} \|f_y(y)\|_{L^2(\Omega)} + \|g(0)\|_{L^2(\Omega)} + \max_{-1 \leq y \leq 0} \|g_y(y)\|_{L^2(\Omega)} + \|\varphi\|_{L^2(\Omega)}\right],
\]

\[
\max_{-1 \leq y \leq 1} \|v(y)\|_{W^1_2(\Omega)} \leq M \left[ \|f(0)\|_{L^2(\Omega)} + \max_{0 \leq y \leq 1} \|f_y(y)\|_{L^2(\Omega)} + \|g(0)\|_{L^2(\Omega)} + \max_{-1 \leq y \leq 0} \|g_y(y)\|_{L^2(\Omega)} + \|\varphi\|_{W^1_2(\Omega)}\right],
\]

\[
\max_{-1 \leq y \leq 1} \|v(y)\|_{W^2_2(\Omega)} \leq M \left[ \|\varphi\|_{W^2_2(\Omega)} + \|f(0)\|_{W^1_2(\Omega)} + \|f_y(0)\|_{L^2(\Omega)} + \max_{0 \leq y \leq 1} \|f_{yy}(y)\|_{L^2(\Omega)} + \|g(0)\|_{W^1_2(\Omega)} + \max_{-1 \leq y \leq 0} \|g_y(y)\|_{L^2(\Omega)} + \max_{0 \leq y \leq 1} \|g_{yy}(y)\|_{L^2(\Omega)}\right].
\]
where $M$ does not depend on $f(y, x)$ ($y \in [0, 1], x \in \overline{\Omega}$), $g(y, x)$ ($y \in [-1, 0], x \in \overline{\Omega}$) and $\varphi(x)$ ($x \in \overline{\Omega}$).

The proof of this theorem is based on the abstract Theorem 2.1 and the symmetry properties of the space operator generated by the problem (3.2).

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